Intersection Pairings on Spaces of Connections and Chern-Simons Theory on Seifert Manifolds

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1 Introduction

Let M be a compact closed 3-manifold and $G \to P \to M$ a trivial principal G bundle over M. We denote by $\mathcal{A}_3$ the space of connections on P and $\mathcal{G}_3$ the gauge group. Let $Z_{k,G}[M]$ denote the Chern-Simons path integral at level k and for group G (taken as above),

$$Z_{k,G}[M] = \frac{1}{\mathrm{Vol}(\mathcal{G}_3)} \int_{\mathcal{A}_3} \exp\left(I(\mathscr{A})\right)$$

where the Chern-Simons action, for $\mathscr{A} \in \mathcal{A}_3$, is

$$I(\mathscr{A}) = i\frac{k}{4\pi} \int_M \mathrm{Tr}\left(\mathscr{A} \wedge d\mathscr{A} + \frac{2}{3}\mathscr{A} \wedge \mathscr{A} \wedge \mathscr{A}\right)$$

and Tr is normalized so that under large gauge transformations (recall that $\pi_0(\mathcal{G}_3) = \mathbb{Z}$) $I(\mathscr{A}^{\mathscr{G}}) = I(\mathscr{A}) + 2\pi i n$, $\mathscr{G} \in \mathcal{G}_3$ and $n \in \mathbb{Z}$ so that the exponential is invariant.

One can also consider the inclusion of 'Wilson lines' in the path integral. A Wilson line is a combination of a knot K and an irreducible representation R of the group G, and is defined to be the holonomy

$$W_R(K) = \mathrm{Tr}_R\, P \exp \oint_K \mathscr{A}$$

The relevant path integral is

$$Z_{k,G}[M, (K_i, R_i)] = \frac{1}{\mathrm{Vol}(\mathcal{G}_3)} \int_{\mathcal{A}_3} \exp\left(I(\mathscr{A})\right) \prod_{i=1} W_{R_i}(K_i)$$

Let Σ be a smooth genus g curve and ω a unit volume Kähler form on Σ. Let $\mathfrak{M}$ be the moduli space of flat G connections on Σ. The moduli space has a natural symplectic, infact Kähler, structure and we let $\mathfrak{L} \to \mathfrak{M}$ be the fundamental line bundle whose first Chern class agrees with the natural symplectic form on $\mathfrak{M}$ (for $G = SU(n)$ this will be the determinant line bundle). E. Witten [15] has shown that the quantum Hilbert space of states of Chern-Simons theory on Σ is $\mathrm{H}^0(\mathfrak{M}, \mathfrak{L}^k)$. Quite generally the Chern-Simons invariant, $Z_{k,G}[M]$, for a 3-manifold M, by Heegard splitting, will be the inner product of two vectors in the Hilbert space. However, the dimension of the Hilbert space is the Chern-Simons invariant of the 3-manifold $\Sigma \times S^1$.

If one includes Wilson lines, then the Hilbert space is $\mathrm{H}^0(\mathfrak{M}, \mathfrak{L}^k)$ where now $\mathfrak{M}$ is the moduli space of parabolic bundles on Σ, which is still naturally Kähler, and $\mathfrak{L}$ is the associated fundamental line bundle.

The Hirzebruch-Riemann-Roch theorem tells us that

$$\sum_{q=0} (-1)^q \dim \mathrm{H}^q(\mathfrak{M}, \mathfrak{L}^k) = \int_{\mathfrak{M}} \mathrm{Todd}(\mathfrak{M}) \wedge \mathrm{Ch}(\mathfrak{L}^k)$$

If the canonical bundle of $\mathfrak{M}$ is negative, as will be the case in the examples we discuss, then the higher cohomology groups are trivial by Kodaira vanishing and we have

$$\dim \mathrm{H}^0(\mathfrak{M}, \mathfrak{L}^k) = \int_{\mathfrak{M}} \mathrm{Todd}(\mathfrak{M}) \wedge \mathrm{Ch}(\mathfrak{L}^k)$$

E. Verlinde [14] provides us with a concrete formula for the dimension of $\mathrm{H}^0(\mathfrak{M}, \mathfrak{L}^k)$. However, as we have seen, Chern-Simons theory provides us with another formulation of E. Verlinde's dimension count, namely

$$Z_{k,G}[\Sigma \times S^1, (K_i, R_i)] = \int_{\mathfrak{M}} \mathrm{Todd}(\mathfrak{M}) \wedge \mathrm{Ch}(\mathfrak{L}^k)$$

The moduli spaces, which thus far have been generically denoted by $\mathfrak{M}$, are singular at the points where there are reducible connections. Nevertheless, suitably interpreted, the index theorem still yields a topological expression for the invariants $Z_{k,G}[\Sigma \times S^1, (K_i, R_i)]$. This raises a

Question: Are there other 3-manifolds whose Chern-Simons invariants, or parts of them, can be expressed as intersection pairings on an appropriate moduli space $\mathfrak{M}$?

The question has been partially answered in the affirmative by Beasley and Witten [4]. From now on set $M \equiv M_{(g,p)}$ to denote the Seifert manifold that is presented as a degree $-p$, $U(1)$ bundle over a Riemann surface Σ of genus g. Using non-Abelian localization for the Seifert 3-manifolds M Beasley and Witten are able to show that (equation (5.176) in their paper with $n = -p$ due to a different choice of orientation and noting that there is a slightly different normalization of Θ),

Proposition 1.1. (Beasley-Witten) Let M be as above. The portion of the Chern-Simons invariant which is localized on the smooth part, $\mathfrak{M}$, of the moduli space of Yang-Mills connections is

$$Z_{k,\,G}[M]|_{\mathfrak{M}} = \frac{1}{|\Gamma|} \exp\left(i\frac{\pi}{2}\eta_0\right) \int_{\mathfrak{M}} \mathrm{Todd}(\mathfrak{M}) \wedge \mathrm{Ch}(\mathfrak{L}^k) \wedge \exp\left(-i\frac{p}{2\pi}(k + c_{\mathbf{g}})\Theta(\mathfrak{M})\right)$$

In this formula, $\Theta(\mathfrak{M})$ is a certain degree 4 cohomolgy class on $\mathfrak{M}$, $c_{\mathbf{g}}$ and Γ are the dual Coxeter number and centre of G respectively while η_0 is the framing of M. Notice that this formula is quite analogous to the Hirzebruch-Riemann-Roch formula and reduces to it when $p = 0$.

Let $\mathcal{A}$ be the space of connections on the trivial G bundle on Σ and $\mathcal{G}$ the associated group of gauge transformations. I will show that

Proposition 1.2. The Chern-Simons path integral on M, $Z_{k,\,G}[M]$, is equal to a path integral on the space of connections over Σ, namely

$$Z_{k,\,G}[M] = \exp\left(i\frac{\pi}{2}\eta_0\right) \frac{1}{\mathrm{Vol}(\mathcal{G})} \int_{\mathcal{A}} \mathrm{Todd}(\mathcal{A}) \wedge \mathrm{Ch}(\mathfrak{L}^k) \wedge \exp\left(-i\frac{p}{2\pi}(k + c_{\mathbf{g}})\Theta(\mathcal{A})\right)$$

in this formula all the classes have the same form as those in Proposition 1.1.

As there is a striking resemblance between the formulae presented in Propositions 1.1 and 1.2 it is worthwhile, at this point, to make some remarks.

Remark 1.3. The formula in Proposition 1.2 is an exact expression for $Z_{k,\,G}[M]$ unlike that in Proposition 1.1 which is a part of the answer.

Remark 1.4. It is quite straightforward to evaluate the path integral in Proposition 1.2. This has been done for general Σ in [8] by Abelianization and for $\Sigma = S^2$ in [4] by non-Abelian localization. The Reshetikhin-Turaev-Witten invariants for Seifert 3-manifolds may be obtained by surgery prescriptions as in [10] and [11] and agree with the path integral results. However, I leave the path integral 'un-integrated' as it brings the geometry to the fore.

Remark 1.5. I am not at all implying that Proposition 1.1 follows easily from Proposition 1.2, though one would reasonably expect that it does. Presumably an application of non-Abelian localization to the path integral appearing in Proposition 1.2 is what is required.

Remark 1.6. Proposition 1.2 has already been established, somewhat indirectly in [1] and more directly in [8], though the language is somewhat different and the classes were not identified in either of these works and so the geometric significance of the right hand side of Proposition 1.2 was not appreciated.

I will prove a rather more general result involving knots in the fibre direction which are located at points $x_i \in \Sigma$ on the base of the fibration and which run along the fibre. The point of view adopted here is that to the parabolic points x_i one 'attaches' a co-adjoint orbit M_{R_i} defined by the representation R_i.

Proposition 1.7. The Chern-Simons path integral on M, with knots in the fibre direction at the points $x_i \in \Sigma$ and associated representations R_i, $Z_{k,\,G}[M,(x_i,R_i)]$, is given by

$$\begin{aligned} Z_{k,G}[M,(x_i,R_i)] &= \exp\left(i\frac{\pi}{2}\eta_0\right)\frac{1}{\mathrm{Vol}(\mathcal{G})}\int_{\mathcal{A}\times\prod_i M_{R_i}} \widehat{A}(\mathcal{A}\times\prod_i M_{R_i}) \wedge \\ & \quad .\exp\left((k+c_{\mathbf{g}})\Omega(\mathcal{A}) + \sum_i \omega(M_{R_i}) - i\frac{p}{2\pi}(k+c_{\mathbf{g}})\Theta(\mathcal{A})\right) \end{aligned}$$

where $\widehat{A}$ is the A hat genus.

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2 Background and Strategy

While it is clear from the description of the Hilbert space of Chern-Simons theory that the moduli spaces are built in, what is not at all obvious is that Chern-Simons should, in general, have knowledge of the cohomology ring of the moduli space. This paragraph is intended to motivate such a connection. Recall that Witten [16] had established that the topological field theory analogue of Donaldson theory on a curve can be mapped to Yang-Mills theory on the curve. Since the topological field theory is designed to probe the cohomology ring of the moduli space we learn that Yang-Mills theory will do just that. The action of Yang-Mills theory is

$$S(F_A,\psi,\phi) = \frac{1}{4\pi^2}\int_\Sigma \mathrm{Tr}\left(i\phi F_A + \frac{1}{2}\psi\wedge\psi\right) + \frac{\epsilon}{8\pi^2}\int_\Sigma \omega\ \mathrm{Tr}\,\phi^2 \tag{2.1}$$

where A is a connection on P, $\phi \in \Gamma(\Sigma, \mathrm{ad}\ P)$ and ψ is interpreted as a one form on the space $\mathcal{A}/\mathcal{G}$ (in terms of the universal bundle construction described in the next section the elements of the action are all cohomology classes on $\mathcal{A}/\mathcal{G}$).

In [4] Beasley and Witten recall that every 3-manifold has a contact structure κ. Here we use the $U(1)$ bundle structure and the associated nowhere vanishing vector field. (This has the advantage that it corresponds to the obvious structure on $\Sigma \times S^1$.) One may, therefore, decompose connections as

$$\mathscr{A} = A + \kappa \frac{1}{2\pi}\phi, \quad \iota_\kappa A = 0$$

The Chern-Simons action, on choice of the contact structure, is

$$I(A, \phi) = i\frac{k}{4\pi^2}\int_M \left(\pi\kappa \,\mathrm{Tr}\, A\iota_\kappa dA + \kappa \,\mathrm{Tr}\, \phi F_A + \kappa d\kappa \frac{1}{4\pi}\,\mathrm{Tr}\,\phi^2\right) \tag{2.2}$$

and this has some resemblance to (2.1). The first term in (2.2) is the 'time' derivative (in the direction of the vector field dual to κ). There are some differences and perhaps the most glaring is that the term proportional to $\psi \wedge \psi$ is absent. However, a variation on the theme is now available. The structure group of a general 3-manifold is $SO(3)$, however, on fixing the bundle structure the compatible structure group reduces to $SO(2)$ (the structure group of Σ). There is a minimal, so called $N = 1$, supersymmetric version of Chern-Simons theory which is not obviously topological. However, when the structure group is reduced to $SO(2)$ one may 'twist' the $N = 1$ supersymmetric Chern-Simons theory to obtain a manifestly topological theory (up to choice of contact structure). The twisted version has action (2.2) augmented with

$$\frac{k}{8\pi^2}\int_M \kappa\ \mathrm{Tr}\,\psi\wedge\psi, \quad \iota_\kappa\psi = 0$$

The $N = 1$ action becomes

$$I(A, \phi, \psi) = \frac{k}{4\pi^2}\int_M \left(i\pi\kappa\,\mathrm{Tr}\, A\wedge\iota_\kappa dA + \kappa\,\mathrm{Tr}\left(i\phi F_A + \frac{1}{2}\psi\wedge\psi\right) + \frac{i}{4\pi}\kappa d\kappa\,\mathrm{Tr}\,\phi^2\right)$$

so that now the resemblance of the two theories is rather remarkable. The supersymmetry transformations are closely linked to those in [4], equation (3.48),

$$QA = i\psi, \quad Q\psi = -d_A\phi - 2\pi\iota_\kappa dA \tag{2.3}$$

The fact that the fields ψ are free means that partition function and knot invariants of the $N = 1$ and the usual Chern-Simons theory agree.

One possible novelty that there is a new set of observables in the theory namely those that are 3-forms on the moduli space

$$\int_M J_C \wedge \mathrm{Tr}\,\phi.\psi$$

with J_C a de Rham current with delta function support on the 1-dimensional cycle $C \subset M$ is, unfortunately, spoilt by the second part of the transformation of ψ in (2.3).

In any case it would appear that Chern-Simons theory is some supersymmetric quantum mechanics on the moduli spaces of interest. Since supersymmetric quantum mechanics is related to index theory one may reasonably expect a positive response to the question.

The strategy we will follow here is to

1. Identify the objects that appear in the path integral in terms of cohomology classes on the space of connections modulo the gauge group. This is done in Section 3.

2. Next we relate the classes that one obtains from the universal bundle construction to classes associated with the tangent bundle of $\mathcal{A}/\mathcal{G}$. These are denoted by $\mathrm{Todd}(\mathcal{A})$ and $\widehat{A}(\mathcal{A})$ in Section 4 for reasons that will become apparent as we go along.

3. One can now express Witten's localisation of the Yang-Mills path integral on Σ (Proposition 5.1 below) in terms of classes that come from the universal bundle on $\mathcal{A}/\mathcal{G}$ on the one hand and the same classes restricted to $\mathfrak{M}$ on the other.

4. Having established the relationship with the cohomology ring in $\mathcal{A}/\mathcal{G}$ our final task is to show that the integral over $\mathcal{A}_3/\mathcal{G}_3$ reduces to that of certain classes on $\mathcal{A}/\mathcal{G}$ on integrating out sections which are not $U(1)$ invariant.

3 A Universal Bundle

Let P be a principal G bundle over some smooth manifold X, $\mathcal{A}$ the space of connections on P and $\mathcal{G}$ the group of gauge transformations (bundle automorphisms). We have the following universal bundle construction due to Atiyah and Singer [2]. There is an action of $\mathcal{G}$ on P so that we may form the space (this is not smooth unless one makes further assumptions and as it stands it is a stack)

$$\mathcal{Q} = \mathcal{A} \times_{\mathcal{G}} P$$

Now G operates on $\mathcal{Q}$ and infact $\mathcal{Q}$ is itself the total space of a principle bundle

$$\mathcal{Q} \to \mathcal{Q}/G = \mathcal{A}/\mathcal{G} \times X$$

There is a natural connection on $\mathcal{Q}$ and from it we can define a curvature 2-form and then Chern classes, via Chern-Weil theory, for an associated rank n vector bundle $\mathcal{E} = \mathcal{Q} \times_G \mathbb{C}^n \to \mathcal{A}/\mathcal{G} \times X$. Finally we restrict to some (hopefully smooth) moduli space $\mathfrak{M} \subset \mathcal{A}/\mathcal{G}$, for which the above construction makes sense.

Remark 3.1. A detailed description of this bundle and its relationship to topological field theories can be found in [6] chapter 5, especially sections 5.1 and 5.3.

Remark 3.2. The path integral we need to deal with is the one over all of $\mathcal{A}/\mathcal{G}$ and not some smooth finite dimensional subspace $\mathfrak{M}$. Consequently, we will be integrating over the stack.

From now on we take X to be Σ. Decompose the curvature 2-form on $\mathcal{E}$ into its Kunneth components as

$$1 \otimes F_A + \Psi + \Phi \otimes 1 \in \mathrm{H}^2(\mathcal{A} \times \Sigma, \mathrm{Lie}\, G)$$

(If $\mathfrak{M}$ is simply connected we have that $\mathrm{Tr}\,\Psi$, restricted to $\mathfrak{M}$, is cohomologously trivial in which case $c_1(\mathcal{E}) = 1 \otimes c_1(E) + \frac{i}{2\pi}\,\mathrm{Tr}\,\Phi \otimes 1$ where $\mathrm{Tr}\,\Phi \in \mathrm{H}^2(\mathfrak{M})$.) If we fix on $G = SU(n)$ then $c_1(\mathcal{E})$ vanishes and the second Chern class decomposes as

$$\begin{aligned} c_2(\mathcal{E}) &= \frac{1}{4\pi^2}\,\mathrm{Tr}\left(\Phi \otimes F_A + \frac{1}{2}\Psi \wedge \Psi\right) + \frac{1}{4\pi^2}\,\mathrm{Tr}\,\Phi \wedge \Psi + \frac{1}{8\pi^2}\,\mathrm{Tr}\,\Phi^2 \otimes 1 \\ &= \Omega(\mathcal{E}) + \gamma(\mathcal{E}) + \Theta(\mathcal{E}) \end{aligned} \tag{3.1}$$

We have a differential Q on the space $\mathcal{A} \times \Sigma$ which satisfies

$$QA = \Psi, \quad Q\Psi = d_A \Phi, \quad Q\Phi = 0, \quad Q^2 = \mathcal{L}_\Phi$$

so that Ψ is a (basis) one form on $\mathcal{A}$ and

$$(Q - d_A)(1 \otimes F_A + \Psi + \Phi \otimes 1) = 0.$$

Hence associated to Q we have equivariant cohomology on $\mathcal{A}/\mathcal{G} \times \Sigma$ and the Chern classes $c_n(\mathcal{E})$ are Q closed.

Note the second Chern class appears as the action of Yang-Mills theory (2.1) if we make the identifications $\Phi = i\phi$ and $\Psi = \psi$, and we consider these as forms on $\mathcal{A}/\mathcal{G} \times \Sigma$

$$S(F_A, \psi, \phi) = \pi_* \left(\Omega(\mathcal{E}) - \epsilon\Theta(\mathcal{E}) \otimes \omega\right) \simeq \Omega(\mathcal{A}) - \epsilon\Theta(\mathcal{A}) \tag{3.2}$$

where $\pi : \mathcal{A}/\mathcal{G} \times \Sigma \to \mathcal{A}/\mathcal{G}$ is projection onto the first factor.

Remark 3.3. The identification (3.2) shows us that the Yang-Mills action is a Q closed form (of mixed degree). It is also quite clearly not Q exact.

In general for a vector bundle V $c_2(\mathrm{End}\, V) = 2rc_2(V) - (r-1)c_1(V)^2$ and consequently $c_2(\mathrm{End}\, \mathcal{E}) = 2rc_2(\mathcal{E})$. The main interest here will be on the trace free part $\mathrm{End}_0\, \mathcal{E}$ (which in any case has the same second Chern class as $\mathrm{End}\, \mathcal{E}$). Note that the classes on $\mathcal{E}$ are in the 'fundamental' representation while they are taken to be in the 'adjoint' representation on $\mathrm{End}_0\, \mathcal{E}$.

In the rank 2 case, Newstead [12], writes the second Chern Class of his universal bundle as

$$c_2(\mathrm{End}\, U) = 2\alpha \otimes \omega + 4\gamma - \beta \otimes 1 \tag{3.3}$$

Thus to make contact with that work restrict to $\mathfrak{M} \subset \mathcal{A}/\mathcal{G}$ and set,

$$\alpha = \frac{1}{4\pi^2} \int_\Sigma \mathrm{Tr}\, \Psi \wedge \Psi, \quad \beta = -\frac{1}{2\pi^2} \mathrm{Tr}\, \Phi^2, \quad \gamma = \frac{1}{4\pi^2} \mathrm{Tr}\, \Phi \wedge \Psi$$

4 The Todd and the $\widehat{A}$ Genera

In order to express the Todd genus in terms of the classes arising from the universal bundle we follow the approach of Newstead [12] for determining the Pontrjagin class in the rank 2 case. Then we use an observation of Thaddeus [13] to give the $\widehat{A}$ class in terms of the Pontrjagin roots and from there Todd.

The tangent bundle, $\mathrm{T}_\mathfrak{M}$, of $\mathfrak{M}$ is given by

$$\mathrm{T}_\mathfrak{M} \simeq R^1\pi_* \mathrm{End}_0\, \mathcal{E}$$

where R^i denotes the i-th direct image sheaf under the map $\pi : \mathfrak{M} \times \Sigma \to \mathfrak{M}$ onto the first factor. The Grothendieck-Riemann-Roch theorem states that

$$\mathrm{Ch}(\mathrm{T}_\mathfrak{M}) - \mathrm{Ch}(\pi_* \mathrm{End}_0\, \mathcal{E}) = -\pi_* \left(\mathrm{Ch}(\mathrm{End}_0\, \mathcal{E})(1 - (g-1)\omega)\right)$$

For the spaces that we are interested in the direct image sheaf $R^0\pi_* \mathrm{End}_0\, \mathcal{E}$ is trivial so

$$\mathrm{Ch}(\mathrm{T}_\mathfrak{M}) = -\pi_* \left(\mathrm{Ch}(\mathrm{End}_0\, \mathcal{E})(1 - (g-1)\omega\right) \tag{4.1}$$

Denote the complexification of the Lie algebra of G by $\mathbf{g}_{\mathbb{C}}$, the complexification of the Cartan subalgebra by $\mathbf{t}_{\mathbb{C}}$ and the space of roots by $\mathbf{k}$ then

$$\mathbf{g}_{\mathbb{C}} = \mathbf{t}_{\mathbb{C}} \oplus \mathbf{k}$$

We also set $\mathbf{k}_+$ to be the space of positive roots.

Lemma 4.1. The Pontrjagin class of the tangent bundle $P(T_{\mathfrak{M}})$ is given by

$$P(T_{\mathfrak{M}}) = \det{}_{\mathbf{k}}(1 + \mathrm{ad}\,\Phi/2\pi)^{2g-2} = \prod_{\mathbf{k}_+} \left(1 - \left(\frac{\alpha(\Phi)}{2\pi}\right)^2\right)^{2g-2}$$

Proof: Quite generally the only classes that contribute to $\mathrm{Ch}_{2m}(\mathrm{T}_{\mathfrak{M}})$ from (4.1) are powers of Φ, so we have $\mathrm{Ch}_{2m}(\mathrm{T}_{\mathfrak{M}}) = (g-1)\,\mathrm{Tr}_{\mathrm{Ad}}\left(\exp\left(i\Phi/2\pi\right)\right)_{2m}$, and consequently $\mathrm{Ch}(\mathrm{T}_{\mathfrak{M}} \oplus \mathrm{T}^*_{\mathfrak{M}}) = 2(g-1)\,\mathrm{Tr}_{\mathrm{Ad}}\left(\exp\left(i\Phi/2\pi\right)\right)$. This is the same as having the direct sum of $2g-2$ copies of a vector bundle with curvature 2-form $\mathrm{ad}\,\Phi$. The Chern class for one copy is $\det_{\mathbf{g}}(1+i\mathrm{ad}\,\Phi/2\pi) = \det_{\mathbf{k}}(1+i\mathrm{ad}\,\Phi/2\pi)$, the direct sum formula for Chern classes gives us that $c(\mathrm{T}_{\mathfrak{M}} \oplus \mathrm{T}^*_{\mathfrak{M}}) = \prod_{\alpha\in\mathbf{k}_+}(1+(\alpha(\Phi)/2\pi)^2)^{2g-2}$. On the otherhand we have that the Pontrjagin classes are related to Chern classes by $p_m(E) = (-1)^m c_{2m}(E\oplus E^*)$ and this completes the proof. $\square$

Lemma 4.2. The Todd class of the tangent bundle of $\mathfrak{M}$ is

$$\mathrm{Todd}(\mathfrak{M}) = \exp\frac{1}{2}c_1(T_{\mathfrak{M}})\,.\,\left(\frac{\det_{\mathbf{k}}\sin\left(\mathrm{ad}\,\Phi/4\pi\right)}{\det_{\mathbf{k}}\left(\mathrm{ad}\,\Phi/4\pi\right)}\right)^{1-g}$$

Proof: Thaddeus (p147 in [13]) notes that on writing, $P = \prod_{i=1}(1+y_i)$ where the y_i are the Pontrjagin roots, then $\widehat{A} = \prod_{i=1}(\sqrt{y_i}/2/\sinh\sqrt{y_i}/2)$. From Lemma 4.1 the roots are $-(\alpha(\Phi)/2\pi)^2$ and they come with a multiplicity of $2g-2$ so that

$$\widehat{A}(\mathfrak{M}) = \prod_{\alpha\in\mathbf{k}_+}\left(\frac{\sinh\left(i\alpha(\Phi)/4\pi\right)}{i\alpha(\Phi)/4\pi}\right)^{2-2g}$$

and the standard relation between Todd and $\widehat{A}$ completes the proof. $\square$

We can also make use of (4.1) to determine the first Chern class of the moduli space. In this case the term proportional to $(g-1)$ cannot contribute since, as we have seen, it contributes to even classes. We have

$$\mathrm{Ch}(\mathrm{End}_0\mathcal{E}) = -c_2(\mathrm{End}_0\mathcal{E}) + \cdots = -2rc_2(\mathcal{E}) + \ldots \tag{4.2}$$

where the ellipses represent higher degree forms.

Theorem 4.3. (J-M. Drezet and M.S. Narasimhan [9]) The first Chern class of the tangent bundle of $\mathfrak{M}(r,d)$ the moduli space of holomorphic vector bundles of rank r and determinant of degree d is

$$c_1(T\mathfrak{M}(r,d)) = 2\,\mathrm{g.c.d}(r,d)\,\Omega(\mathfrak{M}(r,d))$$

where $\Omega(\mathfrak{M}(r,d))$ is the class of the basic line bundle.

In our case $d=0$ so that $c_1(T\mathfrak{M}) = 2r\Omega(\mathfrak{M})$ and we set, on comparing with (4.2), $\Omega(\mathfrak{M}) = \pi_* c_2(\mathcal{E})$ that is

$$\Omega(\mathfrak{M}) = \frac{1}{4\pi^2}\int_\Sigma\left(i\,\mathrm{Tr}\,\phi F_A + \frac{1}{2}\,\mathrm{Tr}\,\Psi\wedge\Psi\right) \tag{4.3}$$

which is the form one would expect on $F_A = 0$.

5 Intersection Pairings on Moduli Spaces

By comparing to the second Chern class of the universal bundle we see that, quite generally, one should think of $S(F_A, \psi, \phi)$ as a form on the space of connections $\mathcal{A}$. With abuse of notation we denote those classes on $\mathcal{A}$ by the same symbols as those on $\mathfrak{M}$,

$$\frac{1}{\mathrm{Vol}(\mathcal{G})}\int_{\mathcal{A}\otimes\Omega^0(\Sigma,\mathbf{g})} \exp\left(S(F_A,\psi,\phi)\right) \equiv \int_{\mathcal{A}/\mathcal{G}} \exp\left(\Omega(\mathcal{A}) - \epsilon\Theta(\mathcal{A})\right) \tag{5.1}$$

Expressing the path integral in this way hides certain things, like the fact that there is a Gaussian integration over the degree four class or, as the space $\mathcal{A}/\mathcal{G}$ is infinite dimensional, one cannot expand the exponential out to pick the top form term. Indeed a detailed analysis of the path integral shows that some care needs to be exercised in the interpretation of the right hand side (and on the left). Nevertheless expressing the path integral in this way is also very suggestive.

E. Witten [16] shows that this path integral essentially devolves to one on the moduli space,

Proposition 5.1. (E. Witten [16]) The path integral localises onto the moduli space of flat connections,

$$\int_{\mathcal{A}/\mathcal{G}} \mathrm{e}^{\left(\Omega(\mathcal{A}) - \epsilon\Theta(\mathcal{A})\right)} = \int_{\mathfrak{M}} \mathrm{e}^{\left(\Omega(\mathfrak{M}) - \epsilon\Theta(\mathfrak{M})\right)} + \text{ terms non} - \text{analytic in } \epsilon$$

and the non-analytic terms vanish as $\epsilon \to 0$ (provided $\mathfrak{M}$ is not singular). The non-analytic terms arise from higher fixed points of the action, that is, from non-flat solutions to $d_A * F_A = 0$.

The intersection pairings on the moduli space of flat G connections on Σ as presented by Witten [16] agree with those derived by Thaddeus [13] in the rank 2 case and predicts those for higher rank.

6 Supersymmetric Quantum Mechanics and Passing from Chern-Simons to Yang-Mills

In [8] one begins with the Chern-Simons theory, then integrates out modes in the bundle direction, to be left with a theory on the Riemann surface. This Abelian theory is then solved to finally provide one with the invariant for the Seifert manifold. However, it is also pointed out that the same result could be obtained by considering instead Yang-Mills theory on Σ with the inclusion of some observables one of which (equation (6.8) there) written in the current notation is

$$j_{\mathbf{g}}(\phi)^{(1-g)} = \widehat{A}(\mathcal{A})$$

This observation was critical for the present study.

In this section the approach of [8] is followed 'half-way' to the point where we have non-Abelian Yang-Mills theory on Σ. In this way we are able to obtain the classes on $\mathcal{A}/\mathcal{G}$ that one must integrate. I will not repeat the entire calculation but, rather, explain the essential ingredients especially those which go beyond [8]. I should point out that, when needed, I will consider the section ϕ to be momentarily constant on Σ and this will simplify the calculation of the determinants that we will come across presently. After the determinants are calculated ϕ will be allowed to be non-constant once more. The justification for this simplification really comes

from knowing that had we abelianized then ϕ would be forced to be constant on Σ and so consequently we lose nothing in making this asumption.

6.1 Fourier Modes and a Gauge Choice

In order to begin the calculation we note that, as there is a non degenerate S^1 action on M, we may decompose all the sections in terms of characters of that action (a Fourier series). We may therfore write

$$\begin{aligned}\mathcal{A}_3 &= \mathcal{A}\oplus\Omega^0(\Sigma,\mathrm{ad}\,P)\oplus_{n\neq 0}\Omega^1(\Sigma,L^{\otimes -np}\otimes\mathrm{ad}\,P)\oplus_{n\neq 0}\Omega^0(\Sigma,L^{\otimes -np}\otimes\mathrm{ad}\,P)\\ \varphi &= \sum_{n=-\infty}^{\infty}\varphi_n,\quad \iota_\kappa d\varphi_n=-2\pi i n\varphi_n,\quad \varphi_n\in\Omega^*(\Sigma,L^{\otimes -np}\otimes\mathrm{ad}\,P)\end{aligned}$$

where L is the line bundle that defines M.

Now note that there is enough gauge symmetry to impose the condition that the section ϕ is constant in the fibre direction, that is $\iota_\kappa.d\phi=0$, and we do this. Alternatively put, we make the identification,

$$\mathcal{A}_3/\mathcal{G}_3\simeq\left(\mathcal{A}\oplus\Omega^0(\Sigma,\mathrm{ad}\,P)\oplus_{n\neq 0}\Omega^1(\Sigma,L^{\otimes -np}\otimes\mathrm{ad}\,P)\right)/\mathcal{G}$$

There is a caveat here as the 'natural' measures do not coincide since the non-constant components of the section ϕ, in $\Omega^0(\Sigma,L^{\otimes -np}\otimes\mathrm{ad}\,P)$ are tangent vectors to the orbit of $\mathcal{G}_3$. To correct for this mismatch one introduces the Faddeev-Popov ghost determinant, $\Delta_{FP}(\phi)$, which is essentially the ratio of the volume of the orbit to that of the group.

In any case the choice of gauge simplifies the path integral immensely.

6.2 Integrating over non-trivial characters

Our aim is to integrate out all those Fourier modes of fields such that $n\neq 0$. As, by the gauge condition, ϕ has no such modes and the integral over ψ is Gaussian, we concentrate on the integral of the A_n for $n\neq 0$. Note that (with A_0 denoted by A again)

$$I(A,\phi,\psi)=kS(F_A,\psi,\phi)+\Delta I \tag{6.1}$$

where $S(F_A,\psi,\phi)$ is the Yang-Mills action with $\epsilon=ip/2\pi$ and

$$\Delta I=\frac{k}{4\pi^2}\int_\Sigma\sum_{n\neq 0}\mathrm{Tr}\left(A_{-n}\wedge(2\pi n+\mathrm{ad}\,\phi)A_{-n}+\psi_n\wedge\psi_{-n}\right) \tag{6.2}$$

Let

$$\exp i\Gamma(A,\phi)=\int\prod_{n\neq 0}dA_n\,d\psi_n\;\Delta_{\mathrm{FP}}(\phi)\,\exp i\Delta I$$

where the Faddeev-Popov determinant Δ_{FP}, as I mentioned above, takes into account the gauge condition on ϕ.

Definition 6.1. We will say that two gauge invariant functions on $\mathcal{A}$ are equivalent if their integrals over $\mathcal{A}/\mathcal{G}$ agree and we will denote that equivalence by $\simeq$.

Proposition 6.2. The supersymmetric quantum mechanics path integral gives, for ϕ valued in the Cartan subalgebra,

$$\exp i\Gamma(A,\phi) \simeq \exp\left(i\frac{\pi}{2}\eta_0\right)\widehat{A}(i\phi)\wedge\exp\left(i\frac{c_{\mathbf{g}}}{4\pi^2}\int_\Sigma \mathrm{Tr}\,(\phi.F_A+\frac{p}{4\pi}\phi^2\omega)\right)$$

where η_0 is the framing correction.

Proof: As one can see the action is such that the part of the connection $A^{\mathbf{k}}$, only enters in a Gaussian fashion and so may easily be integrated out. The integration gives rise to a determinant which requires regularization (a definition). This calculation has been performed in [8] but includes the constant mode (see (B.23) there) and the ϕ there should be rescaled to $\phi/2\pi$ to agree with the definition here. So on putting back the constant mode contribution in that work and changing the normalization of ϕ we obtain the (square root of the) determinant as

$$\prod_{\alpha\in\mathbf{k}}\left(\frac{\prod_n(2\pi n+i\alpha(\phi)/2\pi)}{i\alpha(\phi)/2\pi}\right)^{1-g}=\prod_{\alpha\in\mathbf{k}_+}\left(\frac{\sin^2 i\alpha(\phi)/4\pi}{(i\alpha(\phi)/4\pi)^2}\right)^{1-g}\equiv j_{\mathbf{g}}(\phi)^{1-g}$$

together with the phase (B.31)-(B.34) in [8]. The integral over those $A^{\mathfrak{t}}$ which are non-constant leads to a simple overall factor in front of the path integral. The integral over the non-constant parts of the symplectic volume give rise to a normalization which compensates that of the connections. The calculation presumes ϕ constant (not just along the fibre) as in the path integral over $\mathcal{A}$ it may be taken to be so. □

There are still 2 issues that we need to deal with:

1. Extend Proposition 6.2 to general sections $\phi\in\Gamma(\Sigma,\mathrm{ad}\,P)$.
2. Make sure that supersymmetry is preserved.

Remark 6.3. Both issues are resolved by recalling a basic tennet of renormalizable field theory: Upon regularising a theory it may be neccessary to add to the Lagrangian local counterterms in order to restore symmetries broken by the choice of regularization.

We begin with the first issue. We know that $\Gamma(\phi)$ is formally gauge invariant,

$$\Gamma(g^{-1}\,\phi\,g)=\Gamma(\phi),\quad g\in\mathcal{G}$$

However, the curvature 2-form F_A that appears in the formula in Proposition 6.2 lies in the Cartan direction, $F_A^{\mathfrak{t}}=dA^{\mathfrak{t}}$, and as it stands the result is only gauge invariant under gauge transformations in the maximal torus, $g\in\mathrm{Map}(\Sigma,T)$. The source for this is that the regularization adopted in [8] was only designed to preserve the Torus invariance.

It is straightforward to check that, in general, the absolute value of the determinant is a function of ϕ^2 and indeed agrees with the function $\widehat{A}(\sqrt{\phi^2})$ so that this is invariant under $\mathcal{G}$. Our difficulty, therefore, rests with the phase and we perform a finite renormalization to put in the complete non-Abelian curvature 2-form which re-instates $\mathcal{G}$ invariance.

This is still not quite the end of the story. The one loop correction is not supersymmetric. Or put another way we have not maintained the original supersymmetry (2.3) at the level of the zero modes, which is now,

$$QA=i\psi,\quad Q\psi=-d_A\phi,\quad Q\phi=0, \tag{6.3}$$

We can add another finite renormalization

$$\exp \frac{c_{\mathbf{g}}}{8\pi^2} \int_{\Sigma} \mathrm{Tr}\, \psi \wedge \psi$$

to correct this.

Proposition 6.4. The gauge invariant and supersymmetric evaluation of the path integral along the fibres of M is

$$\exp i\Gamma(A, \phi, \psi) \simeq \exp\left(i\frac{\pi}{2}\eta_0\right) \widehat{A}(i\phi) \wedge \exp\left(\frac{c_{\mathbf{g}}}{4\pi^2} \int_{\Sigma} \mathrm{Tr}\,(i\phi.F_A + \frac{1}{2}\psi \wedge \psi + i\frac{p}{4\pi}\phi^2\omega)\right)$$

Remark 6.5. With the identifications that $\psi \simeq \Psi$ and $\phi \simeq -i\Phi$ and as $c_{\mathbf{g}} = r$ we have

$$\exp i\Gamma(A, \phi, \psi) \simeq \exp\left(i\frac{\pi}{2}\eta_0\right) \mathrm{Todd}(\mathcal{A}) \wedge \exp\left(-i\frac{p}{2\pi} c_{\mathbf{g}} \Theta(\mathcal{A})\right)$$

The path integral now becomes one over objects defined on the Riemann surface directly and we have established Proposition 1.2. □

7 Wilson Lines and Parabolic Points

How is the picture that we have obtained in the previous sections affected by the inclusion of Wilson lines? Since our manifold is a S^1 fibration there is a special class of knots which are located at point $x \in \Sigma$ on the base of the fibration and which run along the fibre. To such a fibre knot we associate

$$W_R(x) = \mathrm{Tr}_R\, P \exp\left(\int_{S^1} \kappa\phi/2\pi\right) = \mathrm{Tr}_R \exp\left(\phi(x)/2\pi\right)$$

the second equality following from the condition that $\iota_\kappa d\phi = 0$. As the only addition to path integral involves functions without dependence on the fibre, the calculation of the previous section goes through unchanged.

A geometric way in which to add such traces is through the introduction of co-adjoint orbits. Let $\lambda \in \mathbf{g}^*$ ($\mathbf{g}^*$ is the dual of $\mathbf{g}$, however, we identify the two so that an invariant inner product $< f, \phi > \equiv \mathrm{Tr}\, f\phi$, $f \in \mathbf{g}^*$, $\phi \in \mathbf{g}$) then the orbit through λ is

$$M_\lambda = \left\{ g^{-1}\lambda g;\ \forall g \in G \right\}$$

while the stabilizer of λ is

$$G(\lambda) = \left\{ g \in G :\ g^{-1}\lambda g = \lambda \right\}$$

If λ is regular ($\det_{\mathbf{k}}(\mathrm{ad}\,\lambda) \neq 0$) then $G(\lambda) = T$ and we consider this case for now so that $M_\lambda = G/G(\lambda) = G/T$.

The homogeneous space G/T comes equipped with a natural G invariant symplectic 2-form (the Kirillov-Konstant form) Ω_λ given by

$$\Omega_\lambda(X, Y) =< \lambda,\, [X, Y] >= \mathrm{Tr}\,(\lambda\,[X, Y]) \quad X,\, Y \in \mathbf{g}$$

Kirillov tells us that for $\lambda = \Lambda + \rho$ regular, Λ an element of the weight lattice and ρ the Weyl vector then

$$\mathrm{Tr}_\lambda\,(\exp \phi/2\pi) = j_{\mathbf{g}}^{-1/2}(\phi/2\pi) \int_{M_\lambda} \exp\left(i\frac{1}{2\pi} < \lambda, \phi > + \Omega_\lambda\right)$$

Now we see that geometrically we should product in the co-adjoint orbits so consider the space $\mathcal{A}_3 \times \prod_i M_{R_i}$, and we have

$$Z_{k,G}[M,(x_i,R_i)] = \frac{1}{\mathrm{Vol}(\mathcal{G}_3)} \int_{\mathcal{A}_3 \times \prod_i M_{R_i}} \exp\left(I(\mathscr{A})\right) \prod_{i=1} j_{\mathbf{g}}^{-1/2}(\phi(x_i)/2\pi) \exp \omega(M_{R_i})$$

where, in analogy with $\Omega(\mathcal{A})$,

$$\omega(M_{R_i}) = \frac{i}{2\pi} \operatorname{Tr} \lambda_i \phi(x_i) + \Omega_{R_i}$$

We have the following:

Lemma 7.1. (Lemma 8.5 [5]) The equivariant $\widehat{A}$-genus, $\widehat{A}_{\mathbf{g}}(X, G/T)$, of the Riemannian manifold G/T and $j_{\mathbf{g}}^{-1/2}(X)$ represent the same class in equivariant deRham cohomology.

Consequently Proposition 1.7 is proved. $\square$

Remark 7.2. C. Beasley [3] has computed, in the spirit of [4], the localization formula for $Z_{k,G}[M,(x_i,R_i)]|_{\mathfrak{M}}$. This formula agrees with that in Proposition 1.7 when restricted to $\mathfrak{M}$.

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